# Algorithm for finding eigenvalues

Suppose $Av = \lambda v$ for some $v \neq 0$, $\lambda \in \mathbb{R}$

Then: $Av - \lambda v = \vec{0}$

$Av - \lambda I v = \vec{0}$

$(A - \lambda I)v = \vec{0}$

Hence $(A-\lambda I)x = \vec{0}$ has a nonzero solution $x = v$.

So $A - \lambda I$ is not invertible, which implies $\det(A - \lambda I) = 0$.

To find the eigenvalues, solve $\underbrace{\det(A - \lambda I) = 0}$

Characteristic Polynomial

Characteristic Equation

Ex: Find the eigenvalues of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det \begin{bmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{bmatrix}$$

$$= (1-\lambda)(2-\lambda) - 30$$

$$= \lambda^2 - 3\lambda - 28$$

$$= (\lambda - 7)(\lambda + 4)$$

Solving $(\lambda - 7)(\lambda + 4) = 0$ gives $\lambda = -4, 7$

Aside: Computationally, this is a terrible way to find eigenvalues since finding roots of polynomials is very hard for degree $n > 4$. One of the final projects explores iterative ways to find eigenvalues (e.g. via QR factorizations)

Ex: The eigenvalues of a triangular matrix are the diagonal entries

$$\det\left(\begin{bmatrix} u_{11} & * \\ 0 & u_{nn} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det \begin{bmatrix} u_{11} - \lambda & * \\ 0 & u_{nn} - \lambda \end{bmatrix} = (u_{11} - \lambda) \cdots (u_{nn} - \lambda) = 0$$

$\Rightarrow \lambda = u_{11}, \ldots, u_{nn}$

## Algorithm for finding eigenvectors

If $Av=\lambda v$, then $(A-\lambda I)v=\vec{0}$.
So the eigenvectors corresponding to $\lambda$ are the nonzero vectors in $N(A-\lambda I)$.

So **to find eigenvectors corresponding to $\lambda$, determine $N(A-\lambda I)$**

Ex: Find all eigenvectors of $A=\begin{bmatrix}1&6\\5&2\end{bmatrix}$

We know $\lambda=-4, 7$.

$\lambda=-4$: $A-(-4)I=\begin{bmatrix}1+4&6\\5&2+4\end{bmatrix}=\begin{bmatrix}5&6\\5&6\end{bmatrix}$

Eigenvectors corresp to $-4$ are nonzero solns to $\begin{bmatrix}5&6\\5&6\end{bmatrix}x=\vec{0}$

We get $x=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-6/5\,x_2\\x_2\end{bmatrix}$ (Check)

So eigenvectors are of the form $\begin{bmatrix}-6/5\,x_2\\x_2\end{bmatrix}$, $x_2\neq 0$.

$\lambda=7$: Similar Calculation gives $\begin{bmatrix}x_2\\x_2\end{bmatrix}$, $x_2\neq 0$ (Check)

Def: Let $\lambda$ be an eigenvalue of $A$. The **eigenspace of $A$ corresponding to $\lambda$** is $E_\lambda(A)=\{v\in\mathbb{R}^n \mid Av=\lambda v\}$

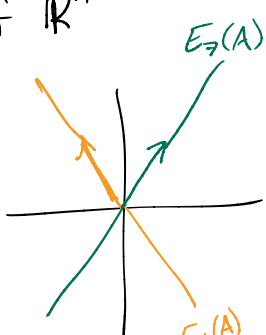
(set of eigenvectors corresp to $\lambda$ along with $\vec{0}$)

Note: $E_\lambda(A)=N(A-\lambda I)$ and thus a subspace of $\mathbb{R}^n$

Ex: Let $A=\begin{bmatrix}1&6\\5&2\end{bmatrix}$.

Then $E_{-4}(A)=\text{Nul}(A+4I)=\text{Span}\left\{\begin{bmatrix}-6/5\\1\end{bmatrix}\right\}$

$E_7(A)=\text{Nul}(A-7I)=\text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$



# Multiplicity

The algebraic multiplicity of $\lambda$ is its multiplicity as a root of the characteristic polynomial.

Ex: $A = \begin{bmatrix} 3 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

Then $\det(A-\lambda I) = (3-\lambda)(3-\lambda)(-1-\lambda)(-1-\lambda)(2-\lambda) = 0$

$\Rightarrow$ $\lambda = 3$ has algebraic multiplicity 2
$\lambda = -1$ has algebraic multiplicity 2
$\lambda = 2$ has algebraic multiplicity 1

The geometric multiplicity of $\lambda$ is $\dim E_\lambda(A)$

Ex: $A = \begin{bmatrix} 3 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

- $E_3(A)$: $A-3I = \begin{bmatrix} 0 & 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & \textcircled{2} & 1 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{-1} \end{bmatrix}$ 3 pivots

Since $(A-3I)\vec{x} = \vec{0}$ has 2 free variables,
$\dim E_3(A) = \dim N(A-3I) = 2$
Hence $\lambda = 3$ has geometric multiplicity 2

- $E_{-1}(A)$: $A+I = \begin{bmatrix} 4 & 1 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$ 4 pivots

Since $(A+I)x=\vec{0}$ has 1 free variable,

$\dim E_{-1}(A) = \dim N(A+I) =$

Hence $\lambda=-1$ has geometric multiplicity 1

- $E_2(A)$: $A-2I = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & 2 & 1 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ 4 pivots

Since $(A-2I)x=\vec{0}$ has 1 free variable,

$\dim E_2(A) = \dim N(A-2I) =$

Hence $\lambda=2$ has geometric multiplicity 1

Note that alg mult of $-1$ $\neq$ geom mult of $-1$

These are different in general.

However, it is true in general that:

$1 \leq$ geom. mult of $\lambda \leq$ alg. mult of $\lambda \leq n$

**Theorem:** $A$ is diagonalizable if and only if it has $n$ linearly independent eigenvectors.

Moreover, if $A$ is diagonalizable, then $A = PDP^{-1}$, where

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad \text{and} \quad P = [v_1 \cdots v_n],$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $A$ and $v_1, \dots, v_n$ are corresponding eigenvectors. (ie. $v_i$ corresponds to $\lambda_i$).

**Proof:**

First note $A = PDP^{-1} \Rightarrow D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $A$ since $A$ and $D$ are similar. Now:

$A$ diagonalizable $\iff A = PDP^{-1}$

$\iff AP = PD$, where $P = [v_1 \cdots v_n]$ is invertible

$\iff [Av_1 \cdots Av_n] = [\lambda_1 v_1 \cdots \lambda_n v_n]$, where $[v_1, \dots, v_n]$ is L.I.

$\iff Av_i = \lambda_i v_i \quad \forall\ 1 \le i \le n$, where $[v_1, \dots, v_n]$ is L.I.

$\iff A$ has $n$ linearly independent eigenvectors

∎

**Key Fact:** If $\{v_1, \dots, v_k\}$ are eigenvectors corresponding to distinct eigenvalues, then $\{v_1, \dots, v_k\}$ is linearly independent